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COMMENT

Critical dynamics of the one-dimensional Potts model

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Abstract. We use the method developed by Achiam and Kosterlitz to obtain the dynamical critical exponent of the one-dimensional q-state Potts model. We find $z=3$ $(q>2)$ and $z=2$ $(q=2)$, thus confirming lower-bound estimates found previously.

In a previous work (Lage 1985, hereafter denoted by I), we introduced a master equation formulation for the kinetic q-state Potts model. We have also shown, using a variational method, that the critical dynamical exponent (z) for the linear chain, with nearest-neighbour couplings, is not smaller than 3 for $q>2$. We now present evidence that $z=3$ for any $q>2$ in the same model.

Let $|\alpha\rangle$ denote the αth state $(\alpha=1,\ldots,q)$ of the spin located at site i (we shall use the notation introduced in I). We denote by $\hat{P}_i^{(\alpha)}$ the projection operator into such a state. The Hamiltonian is, then, written as

$$-\beta H=\sum_i k\bar{P}_i\cdot\bar{P}_{i+1} \tag{1}$$

where k is the nearest-neighbour coupling (measured in units of $k_B T$, where T is the absolute temperature). The equilibrium probability distribution can be written as a state vector in this space:

$$|P_{\text{eq}}\rangle=Z^{-1}\exp(-\beta H)|u\rangle \tag{2}$$

where

$$|u\rangle=\prod_i|u_i\rangle \qquad |u_i\rangle=\sum_{\alpha_i}|\alpha_i\rangle \tag{3a, b}$$

and Z is the partition function:

$$Z=\langle u|\exp(-\beta H)|u\rangle=\operatorname{Tr}\exp(-\beta H).$$

It is easy to decimate over every other spin (e.g., every odd numbered spin), to obtain

$$|P'_{\text{eq}}\rangle=\langle u_{\text{odd}}|P_{\text{eq}}\rangle=Z^{-1}\,\mathrm{e}^{-\beta H'}|u_{\text{even}}\rangle \tag{4}$$

where $|u_{\text{odd}}\rangle$ and $|u_{\text{even}}\rangle$ are defined as in $(3a)$ but with the index running over odd and even integers, respectively. Also, in (4), the transformed Hamiltonian is of the same form as the original one (which shows that decimation is exact), but with a new coupling constant (k'). This can easily be obtained from the identity

$$\operatorname{Tr}_{(i)}\exp[k\bar{P}_{i'}\cdot(\bar{P}_{i-1}+\bar{P}_{i+1})]=\exp[g(k)+f(k)\bar{P}_{i-1'}\cdot\bar{P}_{i+1}] \tag{5}$$

$$k'=f(k)=\log[(q-1+\mathrm{e}^{2k})/(q-2+2\,\mathrm{e}^{k})] \tag{6a}$$

$$g(k)=\log(q-2+2\,\mathrm{e}^{k}). \tag{6b}$$

It is easy to consider the effect of adding a small magnetic field to the original Hamiltonian. If such a term is written as

$$\sum_i \bar{h} \cdot \bar{P}_i$$

(where $\Sigma_\alpha h_\alpha = 0$ can always be imposed, as in I), and treating it as a small perturbation, we find the renormalised fields to be:

$$h'_\alpha = h_\alpha\{1 + 2[(\mathrm{e}^k - 1)/(2\mathrm{e}^k + q - 2)]\}. \tag{7}$$

Under decimation, the new lattice size doubles the original one which leads to the well known result for the correlation length $\xi(k)$

$$\xi(k') = b^{-1}\xi(k) \tag{8}$$

with $b = 2$. This in turn implies that at the non-trivial fixed point ($K = \infty$), the correlation length diverges exponentially:

$$\xi(k) \sim \mathrm{e}^k/q \qquad (k \gg 1).$$

Now, if we start with a non-equilibrium distribution, the relaxation to equilibrium will proceed at a characteristic relaxation rate w (or inverse relaxation time). From the dynamical scaling hypothesis (Halperin and Hohenberg 1969), one has

$$w \propto \xi^{-Z}.$$

Thus, under the above change in length scale, there follows a scaling of the rate

$$w' = b^{-Z}w. \tag{9}$$

This relation will be used to find the dynamical critical exponent (Z).

The master equation governing the time evolution of the equilibrium distribution was studied in I:

$$\frac{\partial}{\partial t}|\mathscr{P}\rangle = \sum_i \left(\sum_r W_r\hat{\psi}_i^{(r)} - \sum_r W_r\right)\hat{M}_i|\mathscr{P}\rangle. \tag{10}$$

Here, the operators $\hat{\psi}_i^{(r)}$ change the states of the spin i:

$$\hat{\psi}_i^{(r)}|\alpha_i\rangle = |\alpha_i - r\rangle \qquad r = 0, 1, \ldots, q-1. \tag{11}$$

Notice that

$$\sum_r \left\{ W_r\hat{\psi}_i^{(r)} - \sum_r W_r \right\}|u_i\rangle = 0. \tag{12}$$

The operator $\hat{M}_i$ was also defined in I:

$$\hat{M}_i = \exp(\beta H)/\mathrm{Tr}_{(i)}\exp(\beta H)$$
$$= [\exp(-k\bar{P}_i \cdot (\bar{P}_{i-1} + \bar{P}_{i+1})]/\mathrm{Tr}_{(i)}\exp(-k\bar{P}_i \cdot (\bar{P}_{i-1} + \bar{P}_{i+1})). \tag{13}$$

Finally, the $q-1$ independent relaxation rates W_r were found in I to obey rather general conditions which we assume here to be satisfied.

We now study the dynamics generated by (1) using the method developed by Achiam and Kosterlitz (1978): after a sufficiently long time, the probability distribution is expected to deviate only slightly from the equilibrium one and it can be written in a form similar to (2) but with time dependent couplings and fields. We shall assume that we can write

$$|\mathscr{P}(t)\rangle = Z^{-1}\exp(-\beta H)\left(1+\sum_i \bar{h}(t)\cdot\bar{P}_i\right)|u\rangle. \tag{14}$$

We shall show that such a form is reproduced by decimation and, therefore, is the appropriate fixed form to choose. The time dependent field satisfies the condition

$$\sum_\alpha h_\alpha(t) = 0.$$

This implies that $Z(t) = Z$ and hence can be ignored. Inserting (14) into (10), decimating over odd spins and using (12), we obtain

$$\frac{\partial}{\partial t}|\mathscr{P}\rangle = \sum_i\left(\sum_r W_r\hat{\psi}_{2i}^{(r)} - \sum_r W_r\right)\left\langle u_{\text{odd}}\middle|\hat{M}_{2i}Z^{-1}\exp(-\beta H)\sum_j \bar{h}(t)\cdot\bar{P}_j\middle|u\right\rangle.$$

To calculate the required trace over odd spins, we notice first that the spin at site $2i$ is decoupled from the other spins. This, together with (12), allows us to write

$$\frac{\partial}{\partial t}|\mathscr{P}\rangle = \sum_i\left(\sum_r W_r\hat{\psi}_{2i}^{(r)} - \sum_r W_r\right)\bar{h}(t)\cdot\bar{P}_{2i}$$
$$\times\frac{\langle u_{\text{odd}}|\hat{M}_{2i}\exp(-\beta H)|u_{\text{odd}}\rangle}{\langle u_{\text{odd}}|\exp(-\beta H)|u_{\text{odd}}\rangle}Z^{-1}\exp(-\beta H')|u_{\text{even}}\rangle \tag{15}$$

where use has been made of (4). This equation is of the same form as (10), with the assumed deviation from equilibrium if we use (7) to substitute for the renormalised field, and provided we show that the factor involving the traces can be written as in (13). This last stage, however, holds true only for $q = 2$ (Ising). For general q, a coupling (in the dynamic matrix, only) of next-nearest neighbours is generated. We thus generalise the definition of $\hat{M}_i$ (equation (13)) to include such couplings

$$\hat{M}_i = \frac{\exp(\beta H)}{\text{Tr}_{(i)}\exp(\beta H)}\cdot\exp(\alpha\bar{P}_{i-1}\cdot\bar{P}_{i+1}). \tag{16}$$

We notice that this new form does not violate any of the required properties of the master equation (as studied in I). Actually, such generalisations of the dynamical matrix have recently been considered for the Ising model (Deker and Haake 1979). Using (16) in (15), after some straightforward but tedious algebra, we obtain (10), with renormalised parameters, namely

$$k' = f(k) \tag{17a}$$

$$\alpha' = f(-f(k)) + \log\frac{q-1+\exp(\alpha+f(k)-f(-k))}{q-2+\exp(f(k))+\exp(\alpha+f(-k))} \tag{17b}$$

$$W'_r = W_r\left(1+2\frac{e^k-1}{2\,e^k+q-2}\right)^{-1}\frac{q-2+\exp(f(k))+\exp(\alpha-f(-k))}{q-2+2\exp(k)}$$
$$\times\frac{q-2+2\exp(-f(k))}{q-2+2\exp(-k)}. \tag{17c}$$

These are our renormalisation group equations which we believe to be exact. The first of these equations (which is the same as (6a)) defines the renormalisation of the equilibrium couplings. The other two equations define the renormalisation of dynamical couplings; in particular, (17*c*) and (9) define the dynamical critical exponent.

We consider here the fixed point results for the dynamical couplings. Particular care must be used to deal with the case $q = 2$ and, therefore, we study this case separate from the others.

(a) $q > 2$. Using (6*a*), we obtain the following asymptotic limits:

$$f(k) \simeq k - \log 2 - \tfrac{1}{2}(q-2)\exp(-k) \qquad (k \gg 1)$$

$$f(k) \simeq \log\frac{q-1}{q-2} - \frac{2}{q-2}\exp(-|k|) \qquad (k < 0;\ |k| \gg 1).$$

Then, from (17*b*), we obtain

$$\alpha^* = f(-\infty) = \log[(q-1)/(q-2)].$$

This in turn allows us to obtain, from (17*c*) at the fixed point,

$$W'_r = \tfrac{1}{8} W_r.$$

Therefore, $z = 3$, consistent with the lower bound found in I.

(b) $q = 2$. In this case, we obtain:

$$f(k) = \log \cosh k.$$

This is now an even function of K, which makes the asymptotic limits to be different from the other cases. We also easily find:

$$\alpha' = \log\frac{1+e^{\alpha}}{2} - \log\frac{e^{\alpha} + \cosh^2 k}{1 + \cosh^2 k}$$

$$W'_r = \frac{W_r}{4}[1 - \tfrac{1}{2}e^{-k}]^{-1}\left(1 + \frac{e^{\alpha}}{\cosh^2 k}\right).$$

The first equation has the obvious stable fixed point solution $\alpha = 0$, which implies the well known result $z = 2$.

We have applied the method developed by Achiam and Kosterlitz (1978) to the one-dimensional, q-state Potts model using the master equation formulation considered in a previous work (Lage 1985). The critical dynamical exponent is found to be equal to 3 for $q > 2$, in agreement with the lower bound estimate found before. The Ising model critical behaviour is therefore different from the other q-state models; this difference arises, essentially, because, in one dimension, the transition takes place at zero temperature.

Consideration of higher dimensionalities is obviously of interest and it is now being investigated.

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